ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Pure and Applied Algebra 195 (2005) 167 – 172

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

# Ratliff–Rush closures of ideals with respect to a Noetherian module

Reza Naghipour

*Department of Mathematics, University of Tabriz, Tabriz, Iran*

Received 12 February 2004; received in revised form 12 May 2004
Communicated by C.A. Weibel
Available online 21 September 2004

**Abstract**

Let $R$ be a commutative Noetherian ring, $E$ a non-zero finitely generated $R$-module and $I$ a $E$-proper ideal of $R$. The purpose of this paper is to provide some new characterizations of when all powers of $I$ are Ratliff–Rush closed with respect to $E$ and to answer a question raised by W. Heinzer et al. in (The Ratliff–Rush Ideals in a Noetherian Ring: A Survey, in Methods in Module Theory, Dekker, New York, 1992, pp. 149–159).


*MSC:* Primary: 13A30, 13E05; secondary: 13A02

## 1. Introduction

Throughout this paper, all rings considered will be commutative and Noetherian and will have non-zero identity elements. Such a ring will be denoted by $R$, and the terminology is, in general, the same as that in [3]. Let $I$ be an ideal of $R$, and let $E$ be a non-zero finitely generated module over $R$. We denote by $\mathscr{R}$ the *Rees ring* $R[u, It] := \bigoplus_{n\in\mathbb{Z}} I^n t^n$ of $R$ w.r.t. $I$, where $t$ is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $E[u, It] := \bigoplus_{n\in\mathbb{Z}} I^n E$ over $\mathscr{R}$ is denoted by $\mathscr{E}$, which is a finitely generated graded $\mathscr{R}$-module. We shall say that $I$ is *E-proper* if $E/IE \neq 0$, and, when this is the case, we define the *E-grade of I* (written *grade*$(I, E)$) to be the maximum length of all $E$-sequences contained in $I$. For any ideal $I$ of $R$, *the radical of I*, denoted by $Rad(I)$, is defined to be the set $\{x \in R : x^n \in I$ for some

*E-mail address:* naghipour@tabriz.ac.ir (R. Naghipour).

0022-4049/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.jpaa.2004.06.008

$n \in \mathbb{N}\}$. A brief summary of the results in this paper will now be given. In [8], Ratliff and Rush studied the interesting ideal,

$$\tilde{I} = \bigcup_{n \geqslant 1} (I^{n+1}:_R I^n) = \{x \in R \,:\, xI^n \subseteq I^{n+1} \text{ for some } n \geqslant 1\},$$

associated with $I$. If *grade* $I > 0$, then this new ideal has some nice properties. For instance,

$$\text{for all sufficiently large } n,\ \widetilde{I^n} = I^n. \tag{1.1}$$

They also proved the interesting fact that, for any $n \geqslant 1$, $\widetilde{I^n}$ is the eventual stable value of the increasing sequence,

$$(I^{n+1}:_R I) \subseteq (I^{n+2}:_R I^2) \subseteq (I^{n+3}:_R I^3) \subseteq \cdots .$$

In [6], a regular ideal $I$, i.e., *grade* $I > 0$, for which $\tilde{I} = I$ is called a *Ratliff–Rush closed ideal*, and the ideal $\tilde{I}$ is called the *Ratliff–Rush ideal associated with* the regular ideal $I$. (For more information about the Ratliff–Rush ideals, see [5,6 and 9].) Subsequently, Heinzer et al. [4] introduced a concept analogous to this for modules over a commutative ring. Let us recall the following definition:

**Definition 1.1** (*see, Heinzer et al. [4]*). Let $R$ be a commutative ring, let $E$ be an $R$-module and let $I$ be an ideal of $R$. *The Ratliff–Rush closure of I* w.r.t. $E$, denoted by $\widetilde{I_E}$, is defined to be the union of $(I^{n+1}E:_E I^n)$, where $n$ varies in $\mathbb{N}$, i.e., $\widetilde{I_E} = \{e \in E \,:\, I^n e \subseteq I^{n+1}E$ for some $n \in \mathbb{N}\}$.

If $E = R$, then the definition reduces to that of the usual Ratliff–Rush ideal associated to $I$ in $R$ (see, [8]). Furthermore, $\widetilde{I_E}$ is a submodule of $E$, and it is easy to see that $IE \subseteq \tilde{I}E \subseteq \widetilde{I_E}$. The ideal $I$ is said to be *Ratliff–Rush closed* w.r.t. $E$ if and only if $IE = \widetilde{I_E}$.

At the end of [4], the authors ask: *What conditions ensure that all suitably high powers of I are Ratliff–Rush closed w.r.t. E*. That is: When does the above condition (1.1) extend to the Ratliff–Rush closure with respect to a module? This is answered in Proposition 2.2(iv).

Let $R$ be a Noetherian ring and $E$ a finitely generated $R$-module. For any ideal $I$ of $R$, we denote by $G_R(I)$ (resp. $G_E(I)$) *the associated graded ring* $\bigoplus_{n \geqslant 0} I^n/I^{n+l}$ (resp. *the associated graded* $G_R(I)$*-module* $\bigoplus_{n \geqslant 0} I^nE/I^{n+1}E$). Heinzer et al. have shown in [4, Fact 9] that there exists an element in the homogeneous ideal $\bigoplus_{n \geqslant 1} I^n/I^{n+1}$ that is a non-zerodivisor on the module $G_E(I)$ if and only if for all positive integers $n$, $\widetilde{I^n_E} = I^nE$. As the main result of Section 3, we characterize, when all powers of an ideal $I$ are Ratliff–Rush closed with respect to $E$ in terms of the associated prime ideals of $G_E(I)$.

## 2. Some properties of Ratliff–Rush closures

In this section, we prove several properties of the Ratliff–Rush closures of powers of ideals with respect to a finitely generated module $E$ over a commutative Noetherian ring $R$. The following lemma, which is a consequence of the Artin–Rees Lemma, is assistant in the proof of 2.2.

**Lemma 2.1.** *Let $R$ be a Noetherian ring and let $E$ be a non-zero finitely generated $R$-module. Suppose that $I$ is an $E$-proper ideal of $R$ such that grade $(I, E) > 0$. Then for all large $n$, $(I^{n+r}E{:}_E I^r) = I^n E$ for every integer $r \geqslant 1$.*

**Proof.** It is enough to show that $(I^{n+1}E{:}_E I) = I^n E$ for all large $n$. To this end, let $x \in I$ be an $E$-regular element and let $l$ be an integer as in [1, Lemma 2.10]. By applying the Artin–Rees Lemma to $xE \subseteq E$, we see that there exists $t > 0$ such that for all large $n$, we have $I^{n+1-t}(I^t E \cap xE) = I^{n+1}E \cap xE$. Now, since $I^{n+1}E \cap xE = x(I^{n+1}E{:}_E x)$, we deduce that $x(I^{n+1}E{:}_E x) \subseteq xI^{n+1-t}E$. It implies that $(I^{n+1}E{:}_E x) \subseteq I^{n+1-t}E \subseteq I^l E$, (note that $x$ is an $E$-regular). Consequently, $I^n E = (I^{n+1}E{:}_E I) \cap I^l E = (I^{n+1}E{:}_E I)$, as desired. □

**Proposition 2.2.** *Let the situation be as in Lemma* 2.1. *Then the following conditions hold*:

(i) *Let $T$ be a Noetherian ring which is a flat $R$-module. Then $\widetilde{I_E} \bigotimes_R T = (\widetilde{IT})_{E \otimes_R T}$. In particular, if $S$ is a multiplicatively closed subset of $R$, then $S^{-1}(\widetilde{I_E}) = (\widetilde{S^{-1}I})_{S^{-1}E}$.*

(ii) *Suppose that $I$ is generated by a quasi-regular $E$-sequence. Then, all powers of $I$ are Ratliff–Rush closed w.r.t. $E$.*

(iii) *Let $n \geqslant 1$ be an integer. Then, $\widetilde{I^n_E}$ is the eventual stable value of the increasing sequence,*

$$(I^{n+1}E{:}_E I) \subseteq (I^{n+2}E{:}_E I^2) \subseteq (I^{n+3}E{:}_E I^3) \subseteq \cdots .$$

(iv) *$\widetilde{I_E} \supseteq \widetilde{I_E}^2 \supseteq \cdots \supseteq \widetilde{I^n_E} = I^n E$ for all large $n$; so that, $I^n$ is Ratliff–Rush closed w.r.t. $E$.*

**Proof.** (i) To prove part (i) use [7, Theorem 18.1] and the definition.

(ii) This follows easily from [3, Theorem 1.1.8] and [4, Fact 9].

(iii) By the definition, we have $\widetilde{I^n_E} = \bigcup_{k \geqslant 1}(I^{n+nk}E{:}_E I^{nk})$. Now, it is readily seen that this the same set as $\bigcup_{k \geqslant 1}(I^{n+k}E{:}_E I^k)$. So (iii) holds.

(iv) For every integer $n \geqslant 1$, we have

$$\bigcup_{k \geqslant 1}(I^{n+k}E{:}_E I^k) \supseteq \bigcup_{k \geqslant 1}(I^{n+k+1}E{:}_E I^k),$$

so that, by (iii), $\widetilde{I^n_E} \supseteq \widetilde{I^{n+1}_E}$. On the other hand, by Lemma 2.1, for all sufficiently large $n$, $(I^{n(k+1)}E{:}_E I^{nk}) = I^n E$ for all integers $k \geqslant 1$. Hence the definition yields that, $\widetilde{I^n_E} = I^n E$ for all large $n$, as desired. □

**Remark 2.3.** Let $A$ be a commutative Noetherian ring and $X$ a finitely generated $A$-module. For an ideal $J$ of $A$ and a submodule $Y \subseteq X$, the increasing sequence of submodules,

$$Y \subseteq (Y{:}_X J) \subseteq (Y{:}_X J^2) \subseteq \cdots \subseteq (Y{:}_X J^n) \subseteq \cdots,$$

becomes stationary. Denote its ultimate constant value by $Y{:}_X\langle J\rangle$. Note that $Y{:}_X\langle J\rangle = Y{:}_X J^n$ for all large $n$.

One has $Ass_A X/(Y:_X\langle J\rangle) = Ass_A(X/Y)\backslash V(J)$. Therefore the primary decomposition of $Y:_X\langle J\rangle$ consists of those primary components of $Y$ whose associated prime ideals do not contain $J$. Now, with the assumption given in Lemma 2.1, it follows from Proposition 2.2 that for every integer $n \geqslant 1$, $(u^n\mathscr{E}:_{\mathscr{E}}\langle It\rangle)\cap E = \widetilde{I^n_E}$.

## 3. Main theorem and its proof

It will be shown in this section that the subjects of the previous section can be used to give a number of characterizations when all powers of an ideal $I$ are Ratliff–Rush closed with respect to a finitely generated module over a commutative Noetherian ring. In fact, six, such characterizations are given in Theorem 3.3.

Following [2], we shall use $A^*(I, E)$ to denote the ultimately constant values of $Ass_R E/I^n E$ for all large $n$.

**Proposition 3.1.** *Let R be a Noetherian ring and E a non-zero finitely generated R-module. Let I and $\mathfrak{p}$ be ideals of R such that $I \subseteq \mathfrak{p} \in Spec\, R$ and grade $(I, E) > 0$. Then,*

(i) *There exists an integer $n \geqslant 1$ such that $\mathfrak{p} \in Ass_R E/I^n E$, if and only if, there exists $\mathfrak{q} \in Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$ with $\mathfrak{q}\cap R = \mathfrak{p}$.*

(ii) *$\mathfrak{p} \in A^*(I, E)$, if and only if, there exists $\mathfrak{q} \in Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$ such that $\mathfrak{q}\cap R = \mathfrak{p}$ and $It \nsubseteq \mathfrak{q}$.*

(iii) *There exists an integer $n \geqslant 1$ such that $\mathfrak{p} \in Ass_R E/I^n E\backslash A^*(I, E)$, if and only if, $\mathfrak{q} := (u, \mathfrak{p}, It)$ is only element of $Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$ such that $\mathfrak{q}\cap R = \mathfrak{p}$.*

**Proof.** (i) Let $n \geqslant 1$ be an integer and $\mathfrak{p} \in Ass_R E/I^n E$. Then, there exists $e \in E$ such that $\mathfrak{p} = (I^n E:_E e)$. It is easy to see that, $\mathfrak{p} = (u^n\mathscr{E}:_{\mathscr{R}} et^0)\cap R$. Now, by [1, Lemma 2.2], there exists $ct^m \in \mathscr{R}$ such that $\mathfrak{q} = (u^n\mathscr{E}:_{\mathscr{R}} cet^m)$ is a prime ideal of $\mathscr{R}$ such that $\mathfrak{q}\cap R = \mathfrak{p}$.

Conversely, suppose $\mathfrak{q} \in Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$ and $\mathfrak{q}\cap R = \mathfrak{p}$. Then, there exists a homogeneous element $et^k \in \mathscr{E}$, $e \in E$, such that $\mathfrak{q} = (u\mathscr{E}:_{\mathscr{R}} et^k)$, by [3, Lemma 1.5.6]. Certainly $k \geqslant 0$, since $et^k \notin u\mathscr{E}$. Accordingly, $\mathfrak{p} = \mathfrak{q}\cap R = (u\mathscr{E}:_{\mathscr{R}} et^k)\cap R = (I^{k+1}E:_R e)$. Hence we obtain $\mathfrak{p} \in Ass_R I^k E/I^{k+1}E$, and so $\mathfrak{p} \in Ass_R E/I^{k+1}E$, as required.

(ii) As $grade(I, E) > 0$, the claim follows easily from [1, Propositions 2.9 and 2.11].

(iii) As $\mathfrak{q} := (u, \mathfrak{p}, It)$ is the largest homogeneous prime ideal in $\mathscr{R}$ such that $\mathfrak{q}\cap R = \mathfrak{p}$, the desired result follows from (i) and (ii). □

The following theorem will serve to shorten the proofs of the main theorem.

**Theorem 3.2.** *Under assumption given in Proposition* 3.1, *the following conditions are equivalent*:

(i) $(u, \mathfrak{p}, It) \in Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$.

(ii) *There exists an integer $n \geqslant 1$ such that $\mathfrak{p} \in Ass_R(I^{n+1}E:_E I)\cap I^{n-1}E/I^n E$.*

(iii) *There exists an integer $n \geqslant 1$ such that $\mathfrak{p} \in Ass_R(I^{n+1}E:_E I)/I^n E$.*

(iv) *There exists an integer $n \geqslant 1$ such that $\mathfrak{p} \in Ass_R \widetilde{I^n_E}/I^n E$.*

**Proof.** (i)⇒(ii) Let $(u, \mathfrak{p}, It) \in Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$. Then, by [3, Lemma 1.5.6], there exists a homogeneous element $et^k \in \mathscr{E}$ such that $(u, \mathfrak{p}, It) = (u\mathscr{E}:_{\mathscr{R}} et^k)$. Clearly, we have $k \geqslant 0$ and $e \in I^k E \backslash I^{k+1}E$. (Note that $et^k \notin u\mathscr{E}$.) As $Iet^{k+1} \subseteq u\mathscr{E}$, we obtain $e \in (I^{k+2}E:_E I) \cap I^k E$. Moreover, $\mathfrak{p} = (u, \mathfrak{p}, It) \cap R = (I^{k+1}E:_R e)$. Hence $\mathfrak{p} \in Ass_R(I^{k+2}E:_E I) \cap I^k E/I^{k+1}E$, as required.

(ii)⇒(iii) Is obvious.

(iii)⇒(iv) As Proposition 2.2 shows that $(I^{k+n}E:_E I^k) = \widetilde{I^n_E}$ for all large *k*, the claim therefore follows.

(iv)⇒(i) Suppose there exists an integer $n \geqslant 1$ such that $\mathfrak{p} \in Ass_R \widetilde{I^n_E}/I^n E$. Then $\mathfrak{p} = (I^n E:_E e)$ for some $e \in \widetilde{I^n_E}$. Consider a minimal primary decomposition $Q_1 \cap Q_2 \cap \cdots \cap Q_s = u^n \mathscr{E}$ in $\mathscr{E}$ such that for each $1 \leqslant i \leqslant s$, $Rad(Q_i :_{\mathscr{R}} \mathscr{E}) = \mathfrak{p}_i$. After an appropriate reordering of the $\mathfrak{p}_i$'s, there will be an integer $t \geqslant 0$ such that $t \leqslant s$ and $It \nsubseteq \mathfrak{p}_i$ for $1 \leqslant i \leqslant t$ and $It \subseteq \mathfrak{p}_j$ for $t+1 \leqslant j \leqslant s$. As $\widetilde{I^n_E} \neq I^n E$, we see that $t < s$, by Remark 2.3. On the other hand, because $Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s\}$, by Proposition 3.1, there exists $1 \leqslant i \leqslant s$ such that $\mathfrak{p}_i \cap R = \mathfrak{p}$. In order to prove (i), it is enough to show that there exists $t+1 \leqslant i \leqslant s$ such that $\mathfrak{p}_i \cap R = \mathfrak{p}$. To this end, suppose the contrary and look for a contradiction. Let *h* be an integer such that $h \leqslant t$ and let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_h$ be the elements of $Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$ such that $\mathfrak{p}_i \cap R = \mathfrak{p}$ for all $i = 1, 2, \ldots, h$. Let $F := Q_1 \cap \cdots \cap Q_h \cap E$ and $L := Q_{h+1} \cap \cdots \cap Q_s \cap E$. Then, by Remark 2.3, $\widetilde{I^n_E} \subseteq F$ and $F \cap L = I^n E$. Hence $e \in F$ and since $\mathfrak{p} = (I^n E:_R e) = (L:_R e)$, it follows that $\mathfrak{p} \in Ass_R E/L$. On the other hand, because of $L = (Q_{h+1} \cap E) \cap \cdots \cap (Q_s \cap E)$ is a primary decomposition for $L$, we see that $Ass_R(E/L) \subseteq \{\mathfrak{p}_{h+1} \cap R, \ldots, \mathfrak{p}_s \cap R\}$. Consequently there exists an integer $h+1 \leqslant j \leqslant s$ such that $\mathfrak{p} = \mathfrak{p}_j \cap R$, and so, we have a contradiction. □

We are now ready to state and prove the main theorem of this section, which gives six characterizations of Ratliff–Rush closed of all powers of an ideal *I* with respect to a finitely generated module over a commutative Noetherian ring. Of these, it is already known that (v) ⇔ (vii), by [4, Fact 9].

**Theorem 3.3.** *Let the situation be as in Proposition* 3.1. *Then the following conditions are equivalent*:

(i) *$It \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in Ass_{\mathscr{R}}\mathscr{E}/u\mathscr{E}$.*
(ii) *$(I^{n+1}E:_E I) = I^n E$ for all integers $n \geqslant 1$.*
(iii) *$(I^{n+1}E:_E I) \cap I^{n-1}E = I^n E$ for all integers $n \geqslant 1$.*
(iv) *There exists an integer $k \geqslant 1$ such that $(I^{n+k}E:_E I^k) = I^n E$ for all integers $n \geqslant 1$.*
(v) *All powers of I are Ratliff–Rush closed w.r.t. E.*
(vi) *There exists an integer $k \geqslant 1$ and an element x in $I^k$ such that $(I^{n+k}E:_E x) = I^n E$ for all integers $n \geqslant 1$.*
(vii) *There exists an element in the homogeneous ideal $\bigoplus_{n \geqslant 1} I^n/I^{n+1}$ of $G_R(I)$ that is a non-zerodivisor on the associated graded module $G_E(I)$.*

**Proof.** The equivalence of statements (i), (ii), (iii), (v) and (vii) follows easily from the Theorem 3.2 and [4, Fact 9]. Next, we show (iv)⇒(v). To do this, in view of Proposition 2.2, it is enough to show that for any $m \geqslant 1$, we have $(I^{n+mk}E:_E I^{mk}) = I^n E$. To see this, we use induction on $m$. When $m=1$, there is nothing to prove. So suppose that $m>1$ and that the result has been proved for smaller values of $m$. Now, by assumption given in (iv) and the induction hypothesis, we have

$$\begin{aligned}(I^{n+mk}E:_E I^{mk}) &= ((I^{n+mk}E:_E I^k):_E I^{(m-1)k}) \\ &= (I^{n+(m-1)k}E:_E I^{(m-1)k}) \\ &= I^n E.\end{aligned}$$

It completes the inductive step.

The implication (ii)⇒(iv) is evident. In order to show that (vi)⇒(iv), for any $n \geqslant 1$, we have $I^nE \subseteq (I^{n+k}E:_E I^k) \subseteq (I^{n+k}E:_E x) = I^nE$, and so (iv) holds.

Finally, in order to complete the proof we have to show that (i)⇒(vi). Assume that (i) holds. Then $(It) \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in Ass_{\mathscr{R}} \mathscr{E}/u\mathscr{E}$. Accordingly, by [3, Lemmas 1.5.6 and 1.5.10], there exists a homogeneous element $xt^k$ in $(It)$ such that $xt^k$ is a non-zerodivisor on $\mathscr{E}/u\mathscr{E}$. Then $x \in I^k$ and $(u^n\mathscr{E}:_{\mathscr{E}} xt^k) = u^n\mathscr{E}$ for all $n \geqslant 1$. (Note that, as $xu^t \in (It)\backslash u\mathscr{E}$, it follows that $k \geqslant 1$.) Consequently, we can deduce that, $I^nE = u^n\mathscr{E} \cap E = (u^n\mathscr{E}:_{\mathscr{E}} xt^k) \cap E = (I^{n+k}E:_E x)$, as required. $\square$

## Acknowledgements

The author is deeply grateful to the referee for very careful reading of the original manuscript and valuable suggestions. This paper was prepared when the author visited as a short visitor of the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. The author would like to thank from the University of Tabriz and ICTP for their financial supports. Also, the author is deeply grateful to Dr. S. Akbari for his careful reading of the first draft.

## References

[1] S.H. Ahn, Asymptotic primes and asymptotic grade on modules, J. Algebra 174 (1995) 980–998.

[2] M. Brodmann, Asymptotic stability of $Ass(M/I^nM)$, Proc. Amer. Math. Soc. 74 (1979) 16–18.

[3] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge University Press, Cambridge, UK, 1993.

[4] W. Heinzer, B. Johnston, D. Lantz, K. Shah, The Ratliff–Rush Ideals in a Noetherian Ring : A Survey, in Methods in Module Theory, Dekker, New York, 1992, pp. 149–159.

[5] W. Heinzer, B. Johnston, D. Lantz, K. Shah, Coefficient ideals in and blowups of a commutative Noetherian domain, J. Algebra 162 (1993) 355–399.

[6] W. Heinzer, D. Lantz, K. Shah, The Ratliff–Rush ideals in a Noetherian ring, Comm. Algebra 20 (1992) 591–622.

[7] M. Nagata, Local Rings, Interscience, New York, 1961.

[8] L.J. Ratliff Jr., D. Rush, Two notes on reductions of ideals, Indiana Univ. Math. J. 27 (1978) 929–934.

[9] M.E. Rossi, I. Swanson, Notes on the Behavior of the Ratliff–Rush Filtration, Commutative Algebra (Grenoble/Lyon, 2001), Contemporary Mathematics, Vol. 331, American Mathematical Society, Providence, RI, 2003, pp. 313–328.